Lecture 6: Vorticity dynamics

In the previous lecture we have seen how rotation can allow for balanced motions, that are nearly steady in time. The force balance for these motions involves a cancellation of the PGF with Coriolis force

$$f\hat{k}\times\vec{u} = -\frac{1}{\rho_0}\nabla p$$

for low Rossby number flows, i.e. the geostrophic balance or for circular vortices with large Ro#s a balance between

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + fu_\theta + \frac{u_\theta^2}{r}$$

the PGF, Coriolis force, & centrifugal force, i.e. the cyclogeostrophic balance.

These relations are useful for calculating the flow field given the pressure field, but they tell us nothing about how these balanced evolve in time. Their evolution is most concisely described in terms of the dynamics of their vorticity

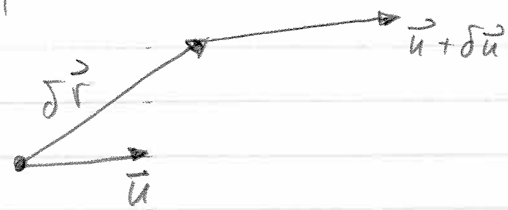
$$\vec{\omega} = \nabla\times\vec{u}$$

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\ ;\quad \omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\ ;\quad \omega_z = \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Looking at the evolution of the vorticity is especially insightful for dynamics of rotating fluids because, as I will show, the vorticity is proportional to the local angular velocity, a spin of the fluid, spin that can be readily inherited from the rotation of the fluid & conservation

To show that the vorticity is proportional to the local spin of the fluid we need to calculate the relative motion near a point:



What is $\delta\vec{u}$ and how does it depend on shear/strain of flow field? Using a Taylor series expansion

$$\vec{u}(\vec{r}+\delta\vec{r}) = \vec{u}(\vec{r}) + \delta\vec{u} = \vec{u}(\vec{r}) + \nabla\vec{u}\cdot\delta\vec{r}$$

$$\delta\vec{u} = \nabla\vec{u}\cdot\delta\vec{r} = \underbrace{\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}}_{\text{velocity gradient tensor} = VGT} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

The VGT can be split into symmetric & antisymmetric parts:

$$\nabla\vec{u} = \frac{1}{2}\overleftrightarrow{S} + \frac{1}{2}\overleftrightarrow{A}$$

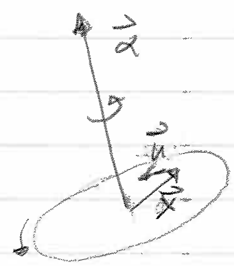
$$\overleftrightarrow{S} = \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{pmatrix} \qquad \overleftrightarrow{A} = \begin{pmatrix} 0 & \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial z}-\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x}-\frac{\partial u}{\partial z} & \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

$$\delta\vec{u} = \nabla\vec{u}\cdot\delta\vec{r} = \frac{1}{2}\overleftrightarrow{S}\cdot\delta\vec{r} + \frac{1}{2}\overleftrightarrow{A}\cdot\delta\vec{r}$$

$$\frac{1}{2}\overleftrightarrow{A}\cdot\delta\vec{r} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_y\delta z - \omega_z\delta y \\ \omega_z\delta x - \omega_x\delta z \\ \omega_x\delta y - \omega_y\delta x \end{pmatrix} = \frac{1}{2}\vec{\omega}\times\delta\vec{r}$$

$$\delta \vec{u} = \frac{1}{2} \overleftrightarrow{S} \cdot \delta \vec{r} + \frac{1}{2} \vec{\omega} \times \delta \vec{r}$$

Focusing on the second term of the relative velocity, you will recognize that if there was a solid body rotating with an angular velocity $\vec{\alpha}$:



Then the velocity at a point $\vec{X}$ away from its axis of rotation will be moving with a velocity
$$\vec{u} = \vec{\alpha} \times \vec{X}$$

It therefore follows that the local angular velocity or spin of the fluid is equal to half the vorticity:

$$\boxed{\frac{1}{2} \vec{\omega} = \text{local spin of fluid} \atop \text{angular velocity}}$$

What processes give rise to a change in vorticity? To answer this question we can construct an equation for the vorticity by taking the curl of the momentum equations:

$$\frac{D\vec{u}}{Dt} + f \hat{k} \times \vec{u} = -\frac{1}{\rho} \nabla p - \nabla \Phi + \vec{F}$$

Note that we have not as yet made the Bous approx

Using a vector identity, one can show that the advective acceleration can be written as:

$$\vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla \left(|\vec{u}|^2 \right) + \vec{\omega} \times \vec{u}$$

$$= \frac{1}{2} \nabla \left(|\vec{u}|^2 \right) + (\nabla \times \vec{u}) \times \vec{u}$$

Vector identity: $\frac{1}{2} \nabla (\vec{A} \cdot \vec{A}) = \vec{A} \times (\nabla \times \vec{A}) + \vec{A} \cdot \nabla \vec{A}$

Using this identity
the momentum equation can be transformed to a vector invariant form (takes same form in all coord syst eqns)

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{2}\nabla\left(|\vec{u}|^2\right) + \vec{\omega}_a \times \vec{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi + \vec{F}$$

Where $\vec{\omega}_a = f\hat{k} + \vec{\omega}$ is the absolute vorticity which is composed of the planetary & relative vorticity.

An equation for the relative vorticity can be constructed by taking the curl of the mom. eqns and noting that
$\nabla \times (\nabla a) = 0$ a is any scalar

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega}_a \times \vec{u}) = -\nabla \times \left(\frac{1}{\rho}\nabla p\right) + \nabla \times \vec{F}$$

Using the vector identities:

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}\,\nabla\cdot\vec{B} + \vec{B}\cdot\nabla \vec{A} - \vec{B}\,\nabla\cdot\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$$

$$\nabla \times (a\vec{A}) = \nabla a \times \vec{A} + a\nabla \times \vec{A}$$

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$$\nabla \times (\vec{\omega}_a \times \vec{u}) = \vec{u}\cdot\nabla\vec{\omega}_a - \vec{\omega}_a\cdot\nabla\vec{u}$$

since $\nabla\cdot\vec{u} = 0$ for an incompressible fluid
$\nabla\cdot\vec{\omega}_a = 0$ since $\nabla\cdot(\nabla\times\vec{A}) = 0$

$$\nabla \times \left(\frac{1}{\rho}\nabla p\right) = -\frac{1}{\rho^2}\nabla\rho \times \nabla p$$

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{u}\cdot\nabla\vec{\omega}_a = \vec{\omega}_a\cdot\nabla\vec{u} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nabla\times\vec{F}$$

Noting that $\partial\vec{\omega}/\partial t = \partial\vec{\omega}_a/\partial t$

$$\boxed{\frac{D\vec{\omega}_a}{Dt} = \vec{\omega}_a\cdot\nabla\vec{u} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nabla\times\vec{F}}$$

Following fluid parcels, $\vec{\omega}_a$ is changed by three processes

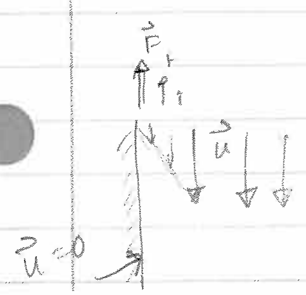
I. Frictional torques $\nabla\times\vec{F}$
II. Baroclinic torque $(1/\rho^2)\nabla\rho\times\nabla p$
III. Vortex stretching/squashing/tilting $\vec{\omega}_a\cdot\nabla\vec{u}$

Describing how these various processes change the vorticity:

I. Frictional torques $\nabla\times\vec{F}$

a) Example 1, frictional forces generated by wind-stress:

A wind-stress will induce a horizontal frictional force that decays in the vertical. If unbalanced it will accelerate a flow in the direction of $\vec{F}$ that similarly decays in the vertical, thus spinning up horizontal vorticity.



Friction induced at a no-slip boundary will generate vertical vorticity

II. Baroclinic torque:

$$\frac{1}{\rho^2} \nabla \rho \times \nabla p$$

For a constant density fluid, this term is zero, also notice that if the flow is <u>barotropic</u> i.e. $p = p(\rho)$ then

$$\nabla p = \frac{dp}{d\rho} \nabla \rho$$

→ pressure & density surfaces coincide

and $\frac{1}{\rho^2} \nabla \rho \times \nabla p = \frac{1}{\rho^2} \frac{dp}{d\rho} \nabla \rho \times \nabla \rho = 0$

A baroclinic flow is one in which density and pressure surfaces do not coincide and hence a baroclinic torque is generated.

The form for the baroclinic torque can be simplified for a Boussinesq fluid, recall for such a fluid:

$$\rho = \rho_0 + \bar{\rho}(z) + \rho'(x,y,z,t)$$

$$p = p_0 + \bar{p}(z) + p'(x,y,z,t)$$

$$\frac{\bar{\rho}}{\rho_0} \sim \frac{\rho'}{\rho_0} \sim \frac{\bar{p}}{p_0} \sim \frac{p'}{p_0} \sim \varepsilon << 1$$

and $0 = -\frac{\nabla p_0}{\rho_0} - g\hat{k}$; $0 = -\frac{\nabla \bar{p}}{\rho_0} - g\frac{\bar{\rho}}{\rho_0}\hat{k}$

Subsituting this form of the density & pressure into the baroclinic torque yields:

$$\frac{1}{\rho^2}\nabla\rho\times\nabla p = \frac{1}{\rho_o^2}\left[1+O(\varepsilon)+\ldots\right]\left[\frac{d\bar\rho}{dz}\hat{k}+\nabla\rho'\right]\times\left[\nabla p_o + O(\varepsilon)+\ldots\right]$$

Thus for $\varepsilon << 1$ the baroclinic torque can be well approximated by:

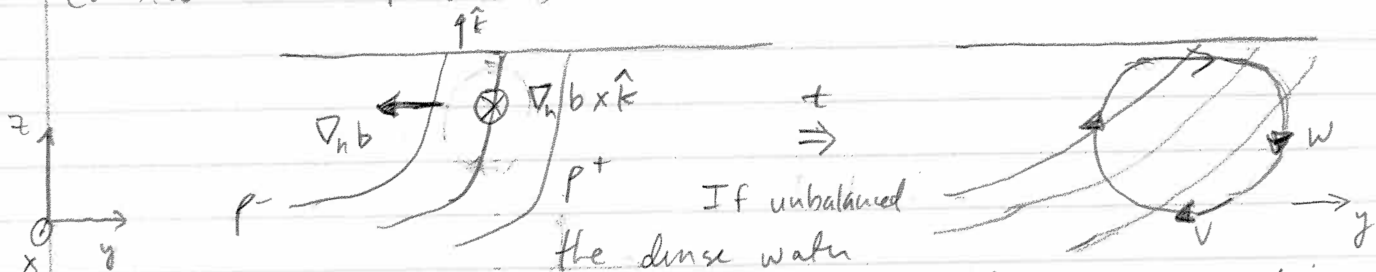
$$\frac{1}{\rho^2}\nabla\rho\times\nabla p \simeq +\frac{1}{\rho_o}\left[\frac{d\bar\rho}{dz}\hat{k}+\nabla\rho'\right]\times\left[-g\hat{k}\right]$$

$$= -\frac{g}{\rho_o}\nabla\rho'\times\hat{k} = \boxed{\nabla b\times\hat{k}}$$

$b = -\frac{g\rho'}{\rho_o}$ is the buoyancy

You will notice that since this involves the cross product with unit vector in the vertical that the baroclinic torque is purely horizontal, i.e. it generates horizontal vorticity. What does it represent?

Consider the following density field:



If unbalanced the dense water would sink under the lighter water generating overturning.

This overturning motion has a horizontal vorticity associated with it

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} < 0$$

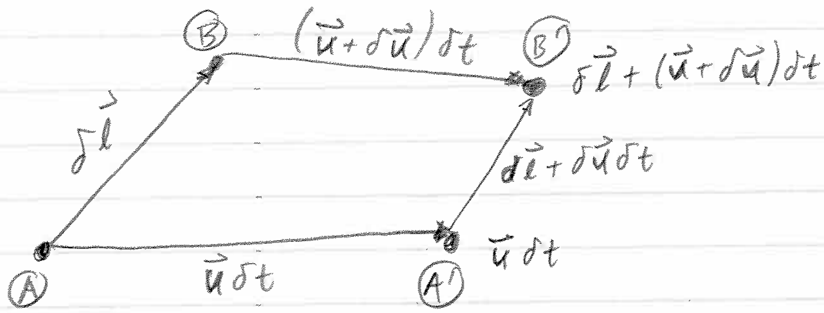
that is generated by the baroclinic torque

III. Vortex stretching / squashing / tilting.

For an inviscid, barotropic fluid, the vorticity evolves via the equation:

$$\frac{D\vec{\omega}_a}{Dt} = \vec{\omega}_a \cdot \nabla \vec{u}$$

As I will show below, the equation governing the evolution of a material line element that marks two fluid parcels separated by small distance takes a form identical to the vorticity equation, and hence will give us the physical insights into what the forcing term represents.



1. Draw a line in the fluid with dye connecting two fluid parcels at points A & B separated by a distance $\delta\vec{l}$.

   The velocity at point A is equal to $\vec{u}$, while at point B it is equal to
   $$\vec{u}+\delta\vec{u} = \vec{u} + \delta\vec{l}\cdot\nabla\vec{u}$$

2. After a short time δt, the parcel at point A will move to point A' at $\vec{u}\delta t$, while ~~the~~ parcel at point B will move to B' at
   $$\delta\vec{l} + (\vec{u}+\delta\vec{u})\delta t$$

The distance between points B' and A' is thus

$$\delta\vec{l} + \delta\vec{u}\,\delta t$$

So the rate of change of the line element following fluid parcels in the limit as $\delta t \to 0$

$$\frac{D\delta\vec{l}}{Dt} = \lim_{\delta t \to 0}\left[\frac{(\delta\vec{l} + \delta\vec{u}\,\delta t - \delta\vec{l})}{\delta t}\right] = \delta\vec{u}$$

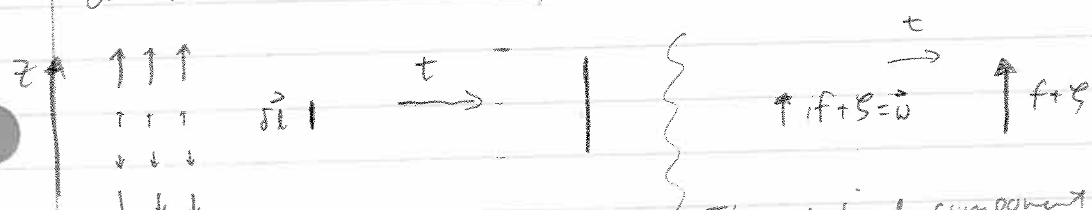
$$\boxed{\frac{D}{Dt}\delta\vec{l} = \delta\vec{l}\cdot\nabla\vec{u}}$$

which takes the identical form of the vorticity eqn:

$$\frac{D}{Dt}\vec{\omega}_a = \vec{\omega}_a\cdot\nabla\vec{u}$$

Therefore to visualize how the term $\vec{\omega}_a\cdot\nabla\vec{u}$ modifies the vorticity we only need to visualize how a small dye line evolves in a given flow field making the analogy between the length of the line element with the magnitude of vorticity and the alignment of the line with the direction of the vorticity.

For example imagine a flow field with a vertical velocity that varies in the vertical:



Put a vertical dye line in flow

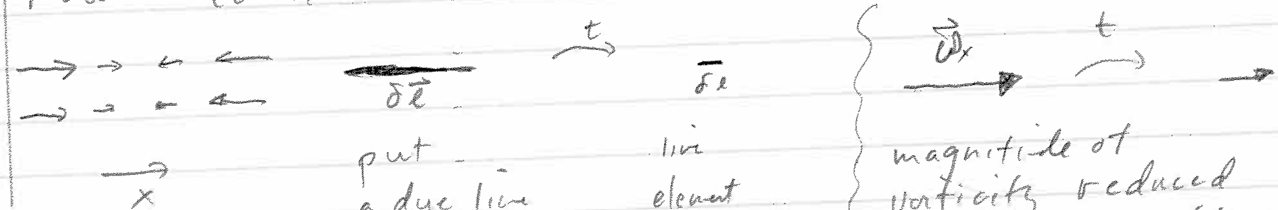
The vertical component of the absolute vorticity will similarly increase in time

The increase in vertical absolute vorticity follows the equation:

$$\frac{D(f+\zeta)}{Dt} = (f+\zeta)\frac{\partial w}{\partial z}$$

The increase in the magnitude of the vorticity by a flow that is divergent in the direction of the absolute vorticity vector is refered to as <u>VORTEX STRETCHING</u>

Now consider a zonal flow that is convergent:



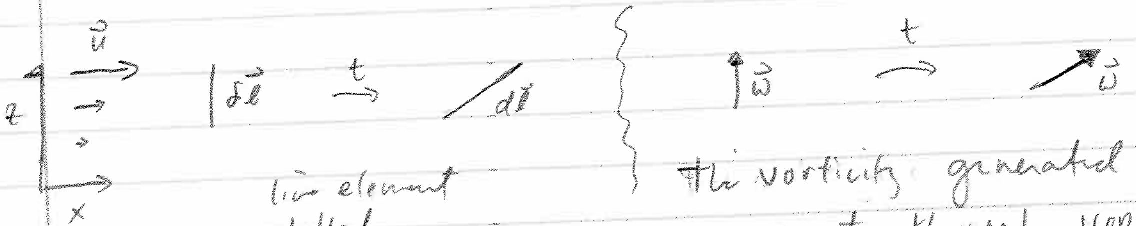
put a dye line horizontaly

line element is squashed

magnitude of vorticity reduced by convergent flow and <u>VORTEX SQUASHING</u>

$D\omega_x/Dt = \omega_x\, du/dx$

Consider a zonal flow that is sheared in the vertical



line element tilted

The vorticity generated a horizontal component through <u>VORTEX TILTING</u>

$$\frac{D\omega_x}{Dt} = \omega_z\frac{\partial u}{\partial z} = (f+\zeta)\frac{\partial u}{\partial z}$$

$$\frac{D\vec{\omega}_a}{Dt} = \vec{\omega}_a \cdot \nabla\vec{u} + \nabla b \times \hat{k} + \nabla \times \vec{F}$$

Consider an inviscid, barotropic fluid (no baroclinic or friction torque) with low Rossby number flow such that $\vec{\omega}_a \simeq f\hat{k}$

Then the scale of

$$\frac{\vec{\omega}_a \cdot \nabla\vec{u}}{D\vec{\omega}_a/Dt} \sim \frac{fU/L}{U/LT} \sim fT \sim \frac{1}{Ro_t} \gg 1$$

$$\sim \frac{fU/L}{(U/L)^2} \sim \frac{fL}{U} \sim \frac{1}{Ro} \gg 1$$

So that in this limit, the vorticity eqn implies:

$$\vec{\omega}_a \cdot \nabla\vec{u} = f\hat{k} \cdot \nabla\vec{u} \simeq 0$$

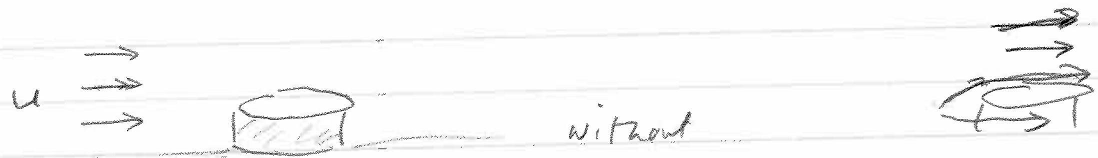
$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial z} \simeq 0$$

The flow has no variations in the direction of the rotation vector.

This is effect is known as the Taylor-Proudman theorem.

⇒ Flows are constrained to be nearly two dimensional because of rotation. Rotation gives the fluid a sense of rigidity.

In the limit of low Ro imagine what happens to a flow around an obstruction in a rotating fluid:



without rotation

what would the flow do as it encounter the object?

→ ① Around the base of the object the flow would be deflected around the sides.

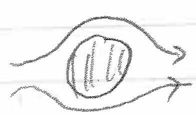
② Just above the object the flow would override the object creating a vertical velocity.

③ Even further above the object the flow would equal the incident flow u.

- Thus the flow would have vertical shear $\partial u/\partial z$, and $\partial w/\partial z \neq 0$.

Now if the fluid were rapidly rotating then the Taylor-Proudman effect kicks in and so $\frac{\partial \vec{u}}{\partial z} = 0$

since $w = 0$ at the bottom, $w = 0$ everywhere
→ the flow can't go over bump.
But the flow must still go around the obstacle. So if $\partial \vec{u}/\partial z = 0$, what does this imply about the flow above the obstacle? show video of Taylor column



Now lets consider a flow with non-zero baroclinicity (i.e. horizontal buoyancy gradients) but in a rotating fluid and assume that the Rossby number of the flow is small. and that the flow is inviscid For these conditions the vorticity equation becomes:

$$0 = f\hat{k}\cdot\nabla\vec{u} + \nabla b \times \hat{k}$$

Which in components becomes:

$$0 = f\frac{du}{dz} + \frac{db}{dy}$$

$$0 = f\frac{dv}{dz} - \frac{db}{dx}$$

$$0 = f\frac{dw}{dz} \rightarrow \frac{\partial u}{\partial x} + \frac{dv}{dy} = 0$$

What do these relations mean? In the first two equations you will recognize that the terms

$f\frac{du}{dz}$ & $f\frac{dv}{dz}$ represent the tilting of the planetary vorticity to the horizontal

So the equation states that the baroclinic torque which would tend to generate horizontal vorticity is counteracted by tilting of planetary vorticity to the horizontal.

An example:

Baroclinic torque would tend to generate vorticity into the page

$\Rightarrow$ what sheared flow would you need to counteract this?

· Hint: recall that a material line element evolves in the same way as the vorticity when acted upon by vortex tilting.

The balance:

$$0 = f\hat{k}\cdot\nabla\vec{u}_h + \nabla_h b \times \hat{k}$$

is known as the THERMAL WIND BALANCE and it is an incredibly useful result. The flow that satisfies this balance is geostrophic (you can convince yourself by vertically integrating the result).

It implies that given observations of the horizontal density gradient one can estimate the velocity field. Oceanographers have been using this result for decades to estimate components of the ocean circulation.

- Show example from Japan/East Sea