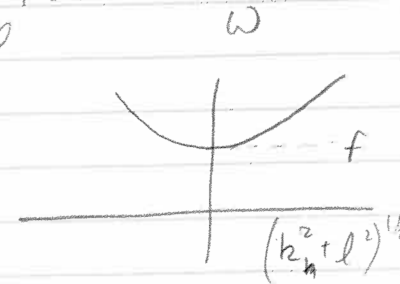
Lecture 5: Balanced flows / motions

From the dispersion relation of inertial gravity waves that we derived last time

$$\omega^2 = \frac{f^2 m^2 + N^2(l^2 + k^2)}{m^2 + l^2 + k^2}$$



we found that inertial-gravity waves with frequencies less than $\omega < f$ cannot exist. So the question arises, what is the physics that governs flows with frequencies $\omega \ll f$, i.e. with timescales

$$\boxed{T \gg \frac{1}{f} \quad \text{subinertial motions}}$$

Or if we define a temporal Rossby number

$$Ro_t = \frac{1}{Tf}$$

flows with $Ro_t \ll 1$?

Oceanic flows typically have timescales set by the advective timescale, i.e.

$$T_{adv} = \frac{L}{U}$$

where L, U are the characteristic length & velocity scales of the flow. Therefore we also would like to know what the dynamics are that govern flows w/

$$Ro = \frac{1}{T_{adv} f} = \frac{U}{fL} \ll 1$$

Let's nondimensionalize the momentum equations to see what ... The dominate force balance is that governs these flows:

$$u = U u' \qquad v = U v' \qquad w = \frac{H}{L} U w'$$

$$p = \rho_0 f U L p' \qquad t = T t' \quad (x, y) = L(x', y')$$

$$z = H z'$$

$$\frac{U}{T} \frac{\partial u'}{\partial t'} + \frac{U^2}{L} \vec{u}' \cdot \nabla u' - f U v' = -f U \frac{\partial p'}{\partial x'}$$

$$\frac{U}{T} \frac{\partial v'}{\partial t} + \frac{U^2}{L} \vec{u}' \cdot \nabla v' + f U u' = -f U \frac{\partial p'}{\partial y}$$

$$\frac{U}{L} \nabla \cdot \vec{u}' = 0$$

Divide mom-eqns by fU, ...

$$Ro_t \frac{\partial u'}{\partial t} + Ro\, \vec{u}' \cdot \nabla u' - v' = -\frac{\partial p'}{\partial x'}$$

$$Ro_t \frac{\partial v'}{\partial t} + Ro\, \vec{u}' \cdot \nabla v' + u' = -\frac{\partial p'}{\partial y'}$$

$$\nabla \cdot \vec{u}' = 0$$

In the limit of $Ro_t \ll 1$ $Ro \ll 1$ the leading order balance becomes:

$$-v' = -\partial p'/\partial x \qquad u' = -\frac{\partial p'}{\partial y}$$

or in dimensional units and in vector form:

$$f\hat{k}\times\vec{u}_g = -\frac{1}{\rho_0}\nabla_h P \qquad \text{Geostrophic balance}$$

$\Rightarrow$ Flow is $\perp$ to the pressure gradient, and the flow that satisfies this balance between the Coriolis force and PGF is refered to as a "geostrophic flow".

If we calculate the horizontal divergence of the geostrophic flow on an f-plane, f = const:

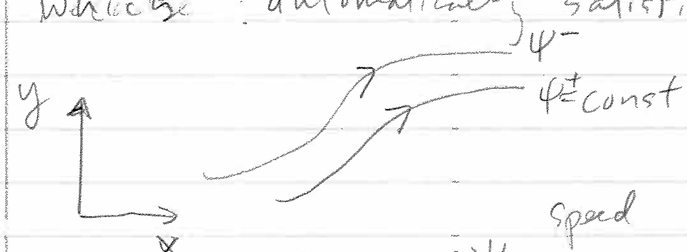
$$\frac{\partial v_g}{\partial y} + \frac{\partial u_g}{\partial x} = \frac{1}{f}\frac{\partial^2 P}{\partial x \partial y} - \frac{1}{f}\frac{\partial^2 P}{\partial x \partial y} = 0$$

we find that the horizontal flow is horizontally non-divergent.

A flow that is horizontally nondivergent can be written in terms of a streamfunction:

$$u_g = -\frac{\partial \psi}{\partial y} \qquad v_g = \frac{\partial \psi}{\partial x}$$

which automatically satisfies $\nabla_h \cdot \vec{u}_g = 0$



Flow is parallel to streamlines and the speed of the flow:

$$(u_g^2 + v_g^2)^{1/2} = |\nabla_h \psi|$$

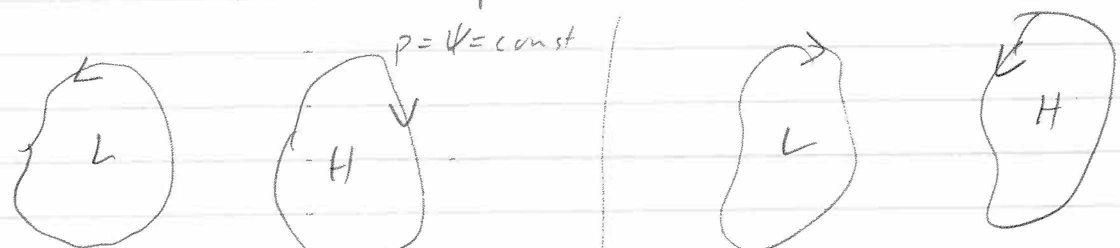
is given by the gradient of the streamfunction.

If we substitute the relationships between u_g, v_g & ψ into the geostrophic balance we find:

$$\psi = \frac{p}{\rho_0 f}$$

Streamlines are coincident with isobars. Geostrophic flow does not move across isobars it moves along them:

The sense of the geostrophic circulation around low & high pressure centers depends on the hemisphere



Northern hemisphere S- hem

For the shallow water equations, the geostrophic balance becomes:

$$f\hat{k} \times \vec{u}_g = -g \nabla_h \eta \quad \Rightarrow \text{ show animations}$$

For an azimuthally symmetric flow this becomes

$$f u_g^{\theta} = g \frac{\partial \eta}{\partial r}$$

What does the geostrophic balance represent physically?

To answer this question it is instructive to go back to the shallow water equations in the non-rotating frame of reference

$$\frac{Dv^{nr}}{Dt} - \frac{(u_\theta^{nr})^2}{r} = -g\frac{\partial \eta^{nr}}{\partial r}$$

$$\frac{Du^{nr}}{Dt} + \frac{u^{nr}v^{nr}}{r} = -\frac{g}{r}\frac{\partial \eta^{nr}}{\partial \theta}$$

Consider the case:

$$v^{nr} = 0 \quad , \quad \underset{\text{az sym}}{\frac{\partial}{\partial \theta} = 0} \quad , \quad \underset{\text{steady}}{\frac{\partial}{\partial t} = 0}$$

$$-\frac{(u_\theta^{nr})^2}{r} = -g\frac{\partial \eta^{nr}}{\partial r}$$

Consider case of a uniformly rotating fluid with an additional azimuthal velocity i.e:

$$u_\theta^{nr} = \Omega r + u_\theta$$

Assume that $u_\theta / \Omega r \ll 1$

$$-\frac{(u_\theta^{nr})^2}{r} \simeq -\Omega^2 r - 2\Omega u_\theta$$

This centrifugal accelation must be balanced by the PGF associated w/ the free surface. If we split the free surface into two parts

$$\eta^{nr} = \bar{\eta} + \eta$$

Then:

$$-\Omega^2 r - 2\Omega u_\theta = -g\frac{\partial}{\partial r}\left(\bar{\eta} + \eta\right) = -g\frac{\partial \eta^{nr}}{\partial r}$$

$$\Rightarrow \quad -\Omega^2 r = -g\frac{\partial}{\partial r}\bar{\eta}$$

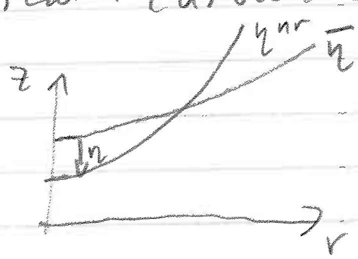
$$f u_\theta = g\frac{\partial \eta}{\partial r}$$

Sketching the solutions:

$\Omega > 0$, $u_\theta > 0$

$\Rightarrow$ centrifugal force greater

$\Rightarrow$ greater curvature of η^{nr}



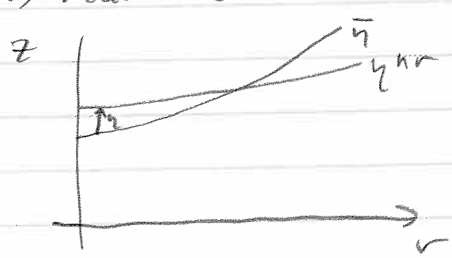
$\eta < 0$ near center

$\Rightarrow$ low pressure center

$\Omega > 0$, $u_\theta < 0$

$\Rightarrow$ centrifugal force reduced

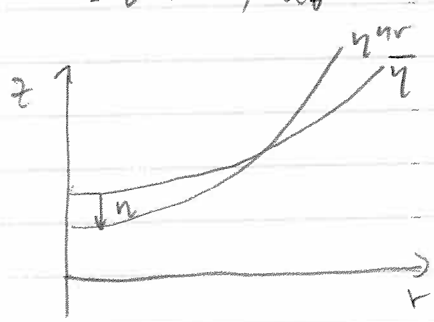
$\Rightarrow$ reduced curvature of η^{nr}



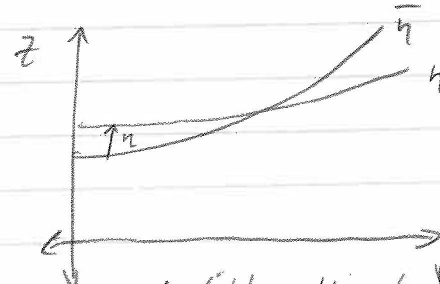
$\eta > 0$ near center

$\Rightarrow$ high pressure center

$\Omega < 0$, $u_\theta < 0$



$\Omega < 0$, $u_\theta > 0$



Note analogy with satellite altimetry SSH cyclonic flows, i.e flows that have the same sense of rotation, i.e. relative vorticity $\nabla \times \vec{u} \cdot \hat{k}$, as the rotation of the fluid, enhance the centrifugal force & hence result in a low pressure center relative to $\bar{\eta}$.

Anticyclonic flows, $\nabla \times \vec{u} \cdot \hat{k} = \zeta$, $\zeta / F < 0$ reduce the centrifugal force & hence result in a high pressure center relative to $\bar{\eta}$.

Sofar we have assumed that

$$Ro \ll 1$$

which for a circular vortex means neglecting higher order contributions to the centrifugal acceleration.
If we relax this restriction then the force balance (in the rotating frame becomes):

$$-fu_\theta - \frac{u_\theta^2}{r} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r}$$

This balance is refered to as the GRADIENT-WIND or CYCLOGEOSTROPHIC Balance.

If we denote $fu_\theta^g = \frac{1}{\rho_0}\frac{\partial p}{\partial r}$

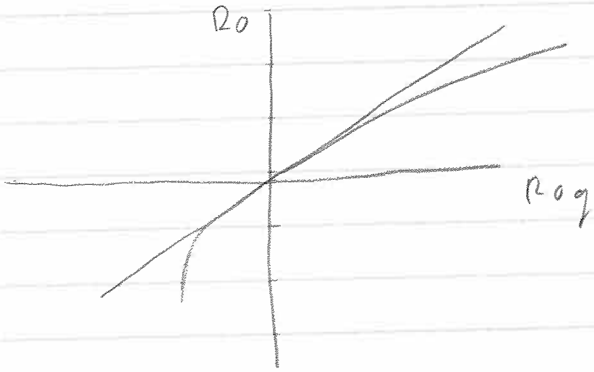
$$-\frac{u_\theta^2}{r} + fu_\theta - fu_\theta^g = 0$$

Dividing through by rf^2 and denoting

$$Ro = \frac{u_\theta}{fr} \qquad Ro^g = \frac{u_\theta^g}{fr}$$

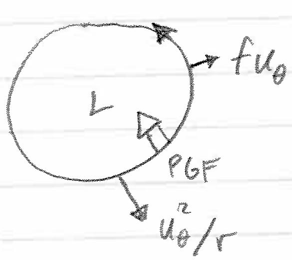
$$Ro^2 + Ro - Ro^g = 0$$

$$\boxed{\begin{aligned} Ro &= -\frac{1}{2} + \sqrt{Ro^g + \frac{1}{4}} \\ u_\theta &= fr\left(-\frac{1}{2} + \sqrt{Ro^g + \frac{1}{4}}\right) \end{aligned}}$$



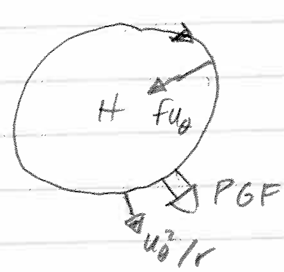
The magnitude of the azimuthal flow is stronger than that predicted by geostrophic balance for anticyclonic flows, weaker for cyclonic flows. Why is this?

CYCLONIC



Centrifugal force opposes PGF $\Rightarrow$ Coriolis force doesn't have to be as strong as in geostrophic balance $\Rightarrow$ $|u_\theta| < |u_\theta^g|$

ANTICYCLONIC



Centrifugal force augments PGF $\Rightarrow$ requires stronger Coriolis force relative to geostrophic

$|u_\theta| > |u_\theta^g|$

Notice that there is a limit to how strong the anticyclonic flow can get, i.e. solutions only exist for

$$Ro^g \geq -\frac{1}{4}$$

What happens at $Ro^g = -\frac{1}{4}$; what does the PGF look like for this Rossby #?

$$u_\theta = -\frac{fr}{2}$$

Going back to the shallow water equations in the non-rotating frame:

$$-\frac{(u_\theta^{nr})^2}{r} = -g\frac{\partial \eta^{nr}}{\partial r}$$

$$u_\theta^{nr} = \Omega r + u_\theta = \Omega r - \frac{fr}{2} = 0$$

$\Rightarrow \frac{\partial \eta^{nr}}{\partial r} \rightarrow$ free surface is flat!
i.e. $\eta^{nr} = \bar{\eta} + \eta = \text{const}$

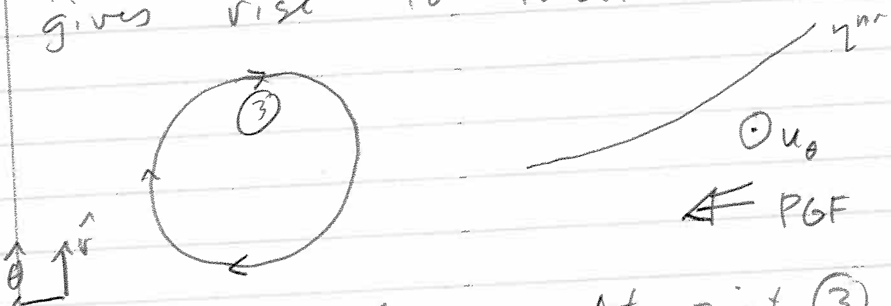
So if $Ro^g = \frac{1}{f^2 r} g \frac{\partial \eta}{\partial r} < -\frac{1}{4}$

then $\frac{\partial \eta^{nr}}{\partial r} < 0$ and the PGF would be outward, in the same direction as the centrifugal force and a steady force balance could not be achieved.

When $u_\theta = -fr/2$, the vertical vorticity of the flow $\zeta = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)$
$\zeta = -f$, and the net vorticity of the fluid $f + \zeta = 0$.

The case where $Ro_g = -\frac{1}{4}$ is an extreme case that shows how the curvature of the free surface is modified by the presence of a balanced flow (cyclonic flows curvature enhanced, anticyclonic flows it is reduced).

Recall that curvature in the free surface plays a critical role in providing the restoring force that gives rise to inertial oscillations:



inertial circle

At point 3 in the inertial circle, the PGF exceeds the centrifugal force & pushes the fluid to lower radius thus providing a restoring force. Therefore for cyclonic (anticyclonic) flows the curvature is greater (weaker) and the restoring force is stronger (weaker) and thus inertial oscillations should have an enhanced (reduced) frequency

Showing this quantitatively:

Consider a 2-D zonal geostrophic flow that varies in the y-direction

$$f u_g = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad u_g = u_g(y)$$

Decompose the flow into geostrophic and ageostrophic components:

$$u = u_g + u'$$

Assume that the flow does not vary in the x-direction $\partial/\partial x = 0$, and is frictionless. The equations of motion become:

$$\frac{Du}{Dt} - fv = 0 \qquad \frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

If we denote Y is the N-S displacement of fluid parcels then

$$v \equiv \frac{DY}{Dt}$$

and:

Absolute momentum!

$$\frac{D}{Dt}(u - fY) = 0 \Rightarrow u - fY = \text{const following fluid parcels}$$

$$\frac{Dv}{Dt} + fu' = 0 \Rightarrow \frac{D^2 Y}{Dt^2} + fu' = 0$$

$$u - fY = u_g + u' - fY = M_0$$

At $t=0$, fluid parcel is at $y = y_0$ and $u = u_g(y_0)$

$M_0 = u_g(y_0) - fy_0$

We can now calculate u' for all time since $u - fY$ is a constant following fluid parcels:

$$u_g(Y) + u' - fY = M_0 = u_g(y_0) - fy_0$$

$$\Rightarrow u'(Y) = u_g(y_0) - u_g(Y) + f(Y - y_0)$$

Denote: $\delta Y = Y - y_0$ assume small displacements

Then we can expand $u_g(Y)$ in a Taylor series about $Y = y_0$:

$$u_g(Y) \approx u_g(y_0) + \left.\frac{\partial u_g}{\partial y}\right|_{y=y_0} (Y - y_0) + \ldots$$

Thus

$$u'(Y) = -\left.\frac{\partial u_g}{\partial y}\right|_{y=y_0} \delta Y + f\,\delta Y$$

$$= \left(f + \zeta_g\big|_{y=y_0}\right)\delta Y$$

Where $\zeta_g = -\left.\frac{\partial u_g}{\partial y}\right|_{y=y_0}$ is the vertical vorticity $\nabla \times \vec{u}_g \cdot \hat{k}$ of the geostrophic flow

Going back to the meridional momentum eqn:

$$\frac{D^2 Y}{Dt^2} + fu' = 0 \quad \Rightarrow \quad \frac{D^2 \delta Y}{Dt^2} + fu' = 0$$

$$\boxed{\frac{D^2}{Dt^2}\delta Y + f(f+\zeta_g|_{y=y_0})\,\delta Y = 0} \Rightarrow$$

Solutions for δY are:

$$\delta Y \sim e^{i\omega t}$$

$$\omega = \sqrt{f(f+\zeta_g|_{y=y_0})}$$

For $\zeta_g + f > 0$ motions are oscillatory $\rightarrow$ inertial oscillations

$\Rightarrow$ For cyclonic flow $\zeta_g / f > 0$ the frequency is greater than f, while for $\zeta_g / f < 0$ the frequency is reduced.

In the GS
We see evidence of the modification of inertial motions by the vorticity of geostrophic flow. Show plots of EMAPEX float data.

Now when $f + \zeta_g < 0$ i.e. $\zeta_g / f < -1$
ω is purely imaginary:

$$\omega = \pm i\sigma \qquad \sigma = \sqrt{f|f+\zeta_g|}$$

$$\delta Y \sim e^{\sigma t},\ e^{-\sigma t}$$

Displacements grow exponentially with time, i.e. such a flow with $\zeta_g / f < -1$ is unstable to small perturbations

$\Rightarrow$ such an unstable flow is refered to being:

CENTRIFUGALLY or INERTIALLY UNSTABLE