

## Lecture 7: Potential Vorticity in the shallow water system and Geostrophic Adjustment ①

In the previous lecture we discussed the dynamics of the vorticity,  $\vec{\omega}$ , whose evolution is governed by the vorticity eqn:

$$\frac{D\vec{\omega}_a}{Dt} = \vec{\omega}_a \cdot \nabla \vec{u} + \nabla b \times \hat{k} + \nabla \times \vec{F}$$

You will recall that the absolute vorticity is equal to

$$\vec{\omega}_a = \frac{1}{2} (\text{angular velocity of the fluid})$$

which is conserved even in the absence of torques:

$$\frac{D\vec{\omega}_a}{Dt} = \vec{\omega}_a \cdot \nabla \vec{u}$$

can change due to vortex stretching/squashing/tilting.


In this lecture, I will introduce a quantity called the potential vorticity that unlike the vorticity is actually conserved in the absence of torques. Knowledge of the PV field can be used to calculate the vorticity and other key quantities of the flow such as the pressure field making it an extremely useful quantity that yields <sup>tremendous</sup> insights into the dynamics of the ocean.

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It is easiest to illustrate the key physics behind the PV using the shallow water equations:

$$\frac{D\vec{u}}{Dt} + f\hat{k} \times \vec{u} = -g\nabla\eta + \vec{F}$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (\vec{u}h) = 0$$

$$h = \eta - b + \#$$


Writing the momentum equation in its vector invariant form:

$$\frac{d\vec{u}}{dt} + (f\hat{k} + \zeta\hat{k}) \times \vec{u} + \frac{1}{2} \nabla_h |\vec{u}|^2 = -g\nabla\eta + \vec{F}$$

and taking the curl ( $\hat{k} \cdot \nabla \times$  (i.e. the vertical component)) the equation for the vorticity becomes:

$$\frac{D}{Dt} (f + \zeta) = -(f + \zeta) \nabla_h \cdot \vec{u} + \hat{k} \cdot \nabla \times \vec{F}$$

From the continuity equation:

$$-\nabla_h \cdot \vec{u} = \frac{1}{h} \frac{Dh}{Dt}$$

Substituting this into the vorticity eqn:

$$\frac{D}{Dt} (f + \zeta) = \frac{(f + \zeta)}{h} \frac{Dh}{Dt} + \hat{k} \cdot \nabla \times \vec{F}$$

Dividing both sides of eqn by  $1/h$ :

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$$\frac{1}{h} \frac{D}{Dt} (f+\zeta) - \frac{(f+\zeta)}{h^2} \frac{Dh}{Dt} = \frac{\hat{\mathbf{k}} \cdot \nabla \times \vec{\mathbf{F}}}{h}$$

Noting that  $\frac{D}{Dt} \frac{1}{h} = -\frac{1}{h^2} \frac{Dh}{Dt}$

$$\frac{1}{h} \frac{D}{Dt} (f+\zeta) + (f+\zeta) \frac{D}{Dt} \left( \frac{1}{h} \right) = \frac{\hat{\mathbf{k}} \cdot \nabla \times \vec{\mathbf{F}}}{h}$$

Using the product rule you see that:

$$\frac{D}{Dt} \left( \frac{f+\zeta}{h} \right) = \frac{\hat{\mathbf{k}} \cdot \nabla \times \vec{\mathbf{F}}}{h}$$

The quantity  $\frac{f+\zeta}{h} = q$  is the potential vorticity in the shallow water system which you will notice that in the absence of frictional torques,  $\nabla \times \vec{\mathbf{F}} = 0$  is conserved following fluid parcels:

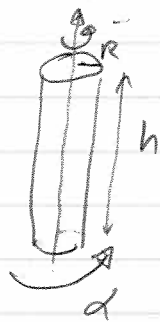
$$\frac{D}{Dt} \left( \frac{f+\zeta}{h} \right) = \frac{Dq}{Dt} = 0$$

Thus unlike the absolute vorticity which is changed by vortex squashing or stretching,  $\frac{Dq}{Dt} = 0$  when  $\nabla \times \vec{\mathbf{F}} = 0$

$$\frac{D}{Dt} (f+\zeta) = - (f+\zeta) \nabla \cdot \vec{\mathbf{u}}$$

The PV is unaffected by this purely inviscid process, it can only be changed by frictional torques.

To understand the physics behind the PV, it is considered a thin column of fluid that always keeps the shape of a cylinder:



- with height  $h$
- radius  $R$
- and angular velocity  $\alpha$

If there are no frictional torques on the fluid cylinder then it will conserve its angular momentum:

$$\frac{d}{dt} (I\alpha) = \text{constant}$$

where  $I = \iiint_{000}^{h \ 2\pi R} r^2 \rho_0 \, r \, dr \, d\phi \, dz = \iiint r^2 \, dm$

is the moment of inertia of the cylinder,

$$I = \rho_0 \pi \frac{R^4}{2} h$$

$$\Rightarrow \rho_0 \pi \frac{R^4}{2} h \alpha = \text{const} \quad \text{Cons of angular momentum}$$

The fluid cylinder must conserve its mass as well:

$$\int \rho_0 \, dV = \text{const}$$

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$$\rho_0 \pi R^2 h = \text{const}$$

$$\Rightarrow R^2 = \frac{G}{h}$$

Substituting this into the expression of angular momentum cons:

$$\rho_0 \frac{\pi}{2} G \frac{\alpha}{h} = \text{const}$$

and remembering that the <sup>absolute</sup> vorticity of the fluid is equal to twice the local angular velocity of the fluid:

$$(F + \gamma) = 2\alpha$$

then the expression for cons of angular momentum becomes

$$\rho_0 \frac{\pi}{4} G \frac{(F + \gamma)}{h} = \text{const}$$

$$\Rightarrow \frac{F + \gamma}{h} = q = \text{const}$$

Showing how PV conservation is simply a consequence of conservation of angular momentum and mass.

Say for example  $h$  increases then the radius of the column must decrease to conserve mass which reduces the moment of inertia & to conserve angular momentum the net spin or abs vorticity must increase. It is the exact same physics that explains why an ice skater rotates faster when they pull their arms in.

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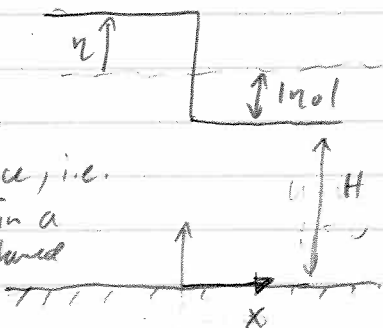
PV conservation is an extremely powerful and useful constraint on the dynamics of oceanic flows and flows in rotating fluids in general.

A particular problem that nicely illustrates the utility of the PV is the process by which a fluid is in an initially unbalanced state ~~naturally~~ evolves to a state of geostrophic balance. This process is known as GEOSTROPHIC ADJUSTMENT and I will illustrate it using the following example.

Consider a shallow layer of water with an initial discontinuity in the free surface but no initial flow:

$$\text{At } t=0 \quad u=v=0$$

At  $t=0$   
the PV is not balanced by Coriolis force, i.e. the fluid is in a completely unbalanced state



$$\eta = \begin{cases} -\eta_0 & x > 0 \\ -\eta_0 & x < 0 \end{cases} = -\eta_0 \operatorname{sgn}(x)$$

Assume that the flow is invariant in the  $y$ -direction  $\partial/\partial y = 0$  and that amplitude of the free surface displacement is small compared to the total depth of the fluid i.e.  $\eta_0 \ll H$ . In this limit the shallow water equations can be linearized and become:

$$\textcircled{A} \quad \frac{\partial u}{\partial t} - fv = -g \frac{\partial \eta}{\partial x} \quad \textcircled{B} \quad \frac{\partial v}{\partial t} + fu = 0$$

$$\textcircled{C} \quad \frac{\partial \eta}{\partial t} + H \frac{\partial u}{\partial x} = 0$$

Taking  $\frac{\partial}{\partial x}$   $\textcircled{A}$  and  $\frac{\partial}{\partial t}$   $\textcircled{C}$

$$\frac{\partial}{\partial t} \frac{\partial u}{\partial x} - f \zeta = -g \frac{\partial^2 \eta}{\partial x^2} \quad \zeta = \frac{\partial v}{\partial x}$$

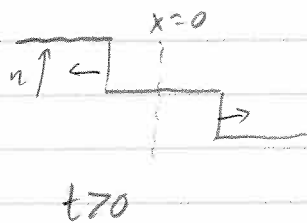
$$\frac{\partial^2 \eta}{\partial t^2} + H \frac{\partial^2 u}{\partial t \partial x} = 0$$

$$\frac{\partial^2 \eta}{\partial t^2} - gH \frac{\partial^2 \eta}{\partial x^2} = -fH \zeta$$

In the limit of no-rotation  $f \rightarrow 0$  this simply becomes the wave eqn:

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \frac{\partial^2 \eta}{\partial x^2} = 0 \quad c^2 = gH$$

Without rotation what will the free surface do with time? Right



Free surface will move up  $x > 0$ , and down for  $x < 0$  as two waves travelling in opposite directions. Showing this quantitatively.

The general solution to the wave equation is

$$\eta = AF(x+ct) + BF(x-ct)$$

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$$\text{At } t=0 \quad \eta = -\eta_0 \operatorname{sgn}(x)$$

$$\Rightarrow -\eta_0 \operatorname{sgn}(x) = (A+B) F(x)$$

$$\text{by symmetry } A=B \quad F = -\frac{\eta_0}{2} \operatorname{sgn}$$

$$\Rightarrow \eta = -\frac{\eta_0}{2} [\operatorname{sgn}(x+ct) + \operatorname{sgn}(x-ct)]$$

The velocity can be calculated from the mom. eqns:

$$\frac{du}{dt} = -g \frac{d\eta}{dx}$$

$$\frac{du}{dt} = -g [A F'(x+ct) + B F'(x-ct)]$$

$$u = \frac{-g}{c} [A F(x+ct) - B F(x-ct)] \quad A+B = -\frac{\eta_0}{2}$$

$$u = +\frac{g\eta_0}{2c} [\operatorname{sgn}(x+ct) - \operatorname{sgn}(x-ct)]$$

$$\eta = \begin{cases} -\eta_0 \operatorname{sgn}(x) & |x| > ct \\ 0 & |x| < ct \end{cases} \quad u = \begin{cases} 0 & |x| > ct \\ g\eta_0/c & |x| < ct \end{cases}$$

This solution is shown in this animation:  
 $\rightarrow$  Go to ppt.

Now what would happen when we add rotation?

① Acceleration of  $\eta$  by Coriolis force ② Balance is achieved

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How strong is the geostrophic flow and what is its spatial structure given the initial step function perturbation in the free surface?

pert is eventually attained

Potential vorticity conservation provides a powerful constraint that allows us to answer this question. Showing this.

Recall that we have assumed that the perturbation to the free surface is very small compared to the thickness of the layer  $\eta_0 \ll H$ . In this limit the statement of PV conservation becomes

$$\frac{Dq}{Dt} \approx \frac{\partial q}{\partial t} = 0$$

$$\left\{ \begin{array}{l} \text{To show this } \propto F^{-1} \\ U \sim g\eta_0/c \quad \vec{u} \cdot \partial q / \partial y \sim \frac{U}{FL} \\ L \sim L_r = c/F \quad \frac{\partial q}{\partial x} / \frac{\partial q}{\partial t} \sim \frac{U}{FL} \\ U/FL \sim g\eta_0/c^2 = \eta_0/H \end{array} \right.$$

$$\text{And } q = \frac{f+s}{H+\eta} \approx \frac{(f+s)}{H} \left(1 + \frac{\eta}{H}\right)^{-1} \approx \frac{(f+s)}{H} \left(1 - \frac{\eta}{H} + \dots\right) \quad \frac{s}{f} \ll 1, \quad \eta/H \ll 1$$

$$q \approx \frac{f}{H} + \frac{s}{H} - \frac{\eta f}{H^2} = \frac{f}{H} \left(1 + \frac{q'}{f} - \frac{\eta}{H}\right) = \frac{s}{H} - \frac{\eta f}{H^2}$$

Thus the statement of PV conservation  $\frac{dq}{dt} = 0$  implies that

$$1 + \frac{s}{H} - \frac{\eta f}{H^2} = \text{constant} = q' + \frac{\eta}{H} \quad \text{for all time}$$

Since  $s=0$  at  $t=0$

Now for long times, i.e. once the flow has reached a steady state, the flow establishes a geostrophic balance:

$$-fV = -g \frac{dn}{dx}$$

and hence the vorticity  $\zeta = \frac{\partial V}{\partial x}$  (i.e. flow is 1-D) is related to free surface by:

$$\zeta = \frac{g}{f} \frac{\partial^2 n}{\partial x^2}$$

Thus for this balanced flow, the free surface is related to the PV field through the following relationship:

$$\frac{g}{fH} \frac{\partial^2 n}{\partial x^2} - \frac{f}{H} n = q'$$

$$\boxed{\frac{\partial^2 n}{\partial x^2} - \frac{1}{L_r^2} n = \frac{fH}{g} q'}$$

where  $L_r = \frac{\sqrt{gH}}{f}$

is known as the Rossby radius of def

The free surface of the balanced flow given a PV field can be found by solving or inverting this equation for  $n$ . Once  $n$  is known, the balanced velocity field can be calculated from geostrophy.

I.E. all dynamically important variables,  $n, v$  can be calculated from the PV  $\rightarrow$  this property of the PV is known as PV INVERTIBILITY

This is an incredibly important and useful result. Illustrating its utility using our example for the problem of geostrophic adjustment.

In this problem, the PV of the initial condition is:

$$q = \frac{f}{H} + \frac{S}{H} - \frac{\eta f}{H^2} = \frac{f}{H} + \frac{\eta_0 f}{H^2} \operatorname{sgn}(x) \quad \text{if } t = 0$$

$$\Rightarrow q' = \frac{\eta_0 f}{H^2} \operatorname{sgn}(x)$$

Thus the equation to solve for the balanced flow is:

$$\textcircled{A} \quad \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{L_r^2} \eta = \frac{\eta_0 f}{L_r^2} \operatorname{sgn}(x) = F(x)$$

We can solve this inhomogeneous ODE using the method of Green's functions.

The Green's function to the above equation satisfies the following eqn:

$$\frac{d^2 G}{dx^2} - \frac{1}{L_r^2} G = \delta(x - x')$$

So that the solution to  $\textcircled{A}$  is:

$$\eta(x) = \int_{-\infty}^{\infty} F(x') G(x, x') dx'$$

showing that this is the solution

$$\begin{aligned}\frac{\partial^2 \eta}{\partial x^2} - \frac{1}{L_r^2} \eta &= \int_{-\infty}^{\infty} F(x') \left[ \frac{d^2 G}{dx^2} - \frac{1}{L_r^2} G \right] dx' \\ &= \int_{-\infty}^{\infty} F(x') \delta(x-x') dx' = F(x) \quad \checkmark\end{aligned}$$

Solving for  $G(x, x')$ :

$$\frac{d^2 G}{dx^2} - \frac{1}{L_r^2} G = \delta(x-x')$$

Recall the definition of the Dirac delta function

$$\delta(x-x') = \begin{cases} 0 & x < x' \\ \infty & x = x' \\ 0 & x > x' \end{cases}$$

Making the equation:

$$\frac{d^2 G}{dx^2} - \frac{1}{L_r^2} G = \begin{cases} 0 & x \neq x' \\ \infty & x = x' \end{cases}$$

For  $x \neq x'$  the soln to the eqn is just

$$G \sim e^{\pm x/L_r}$$

But we want to ensure that  $x \rightarrow \pm \infty \quad G \rightarrow 0$  consequently

$$G = \begin{cases} A e^{x/L_r} & x < x' \\ B e^{-x/L_r} & x > x' \end{cases}$$

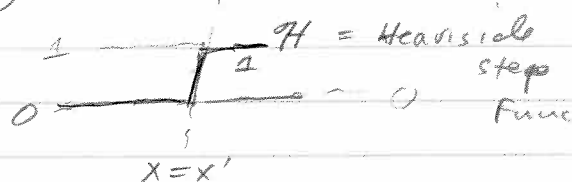
To find  $A, B$ , must impose that the function is cont at  $x=x'$

$$A e^{x'/Lr} = B e^{-x'/Lr}$$

$$A = B e^{-2x'/Lr}$$

At  $x' = x'$  we must have  $\frac{db}{dx}$  be discontinuous, specifically

$$\frac{db}{dx}\bigg|_{x=x'} = H(x-x')$$



Since  $\frac{d^2b}{dx^2}\bigg|_{x=x'} \Rightarrow \infty$

So we make  $\frac{db}{dx}\bigg|_{x=x'+\epsilon} = \frac{db}{dx}\bigg|_{x=x'-\epsilon} + 1$   $\epsilon$  small

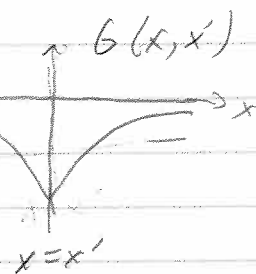
$$\frac{A}{Lr} e^{x'/Lr} + 1 = -\frac{B}{Lr} e^{-x'/Lr}$$

$$\frac{B}{Lr} e^{-x'/Lr} + 1 = -\frac{B}{Lr} e^{-x'/Lr}$$

$$\frac{2B}{Lr} e^{-x'/Lr} = -1 \quad B = -\frac{Lr}{2} e^{x'/Lr}$$

$$A = -\frac{Lr}{2} e^{-x'/Lr}$$

$$G(x, x') = \begin{cases} -\frac{Lr}{2} \exp\left(-\frac{(x-x')}{Lr}\right) & x < x' \\ -\frac{Lr}{2} \exp\left(-\frac{(x-x')}{Lr}\right) & x > x' \end{cases}$$



So

$$\eta(x) = -\frac{Lr}{2} \left[ \int_{-\infty}^x F(x') e^{-\frac{(x-x')}{Lr}} dx' + \int_x^{\infty} F(x') e^{-\frac{(x-x')}{Lr}} dx' \right]$$

$\uparrow$   
 $x > \text{then } x'$

$\uparrow$   
 $x$   
 $\uparrow$   $x$  is always less than  $x'$

Slow ppt slide

$$F(x) = \frac{n_0}{L_r} \operatorname{sgn}(x) = \frac{n_0}{L_r} \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

Substitute this into the solution

For  $x < 0$ :

$$\eta = -\frac{n_0}{2L_r} \left[ \int_{-\infty}^x e^{-(x-x')/L_r} dx' - \int_x^0 e^{(x-x')/L_r} dx' + \int_0^{\infty} e^{(x-x')/L_r} dx' \right]$$

$x > x' \quad x' < 0 \quad x < x' \quad x' < 0 \quad x < x' \quad x' > 0$   
 $\operatorname{sgn}(x') = -1 \quad \operatorname{sgn}(x') = +1 \quad \operatorname{sgn}(x') > 0$

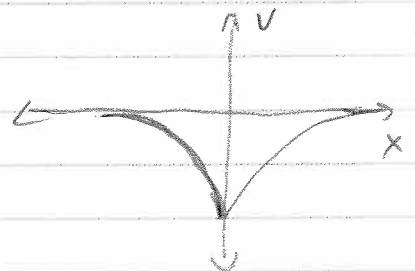
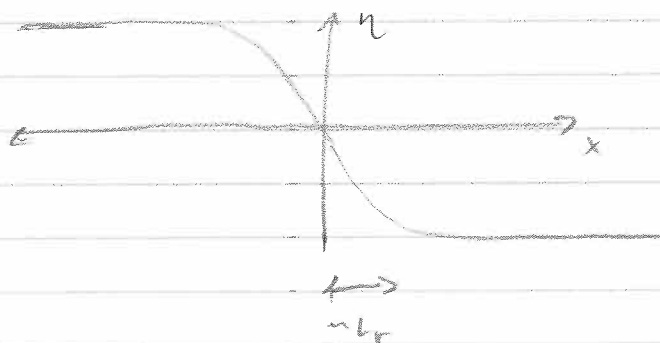
For  $x > 0$ :

$$\eta = -\frac{n_0}{2L_r} \left[ -\int_{-\infty}^0 e^{-(x-x')/L_r} dx' + \int_0^x e^{-(x-x')/L_r} dx' + \int_x^{\infty} e^{(x-x')/L_r} dx' \right]$$

$x > x', x' < 0 \quad x > x', x' > 0 \quad x < x', x' > 0$   
 $\operatorname{sgn}(x') < 0 \quad \operatorname{sgn}(x') > 0 \quad \operatorname{sgn}(x') > 0$

After calculating the integrals the solution becomes:

$$\eta = n_0 \begin{cases} 1 - e^{x/L_r} & x < 0 \\ e^{-x/L_r} - 1 & x > 0 \end{cases} \quad V = -\frac{n_0 g}{f L_r} e^{-|x|/L_r}$$



Notice that the fluid selects a length scale during the adjustment process:

$$L_R = \frac{\sqrt{gH}}{f}$$

i.e. the Rossby radius of deformation.

This is the length scale that is naturally selected by a fluid undergoing adjustment under the effects of both rotation & gravity. What is its physical interpretation?

The Rossby radius is the distance travelled by a gravity wave in a time  $\frac{1}{f}$ . Thus

after this time, the motions of the gravity waves will be able to feel the effects of rotation, i.e. generate a Coriolis force that eventually balances the PGF.

For lengthscales much less than the Rossby radius, the time that it takes for a gravity wave to travel this distance is much less than  $1/f$  and hence their motions are not affected by rotation.

in PE + KE

Energetics of geostrophic adjustment:

The PE per unit volume is  $\rho g z$   
 KE " " "  $\frac{1}{2} \rho (u^2 + v^2)$

Thus for a layer of water with  
 uniform density  $\rho_0$  the total  
 PE is

$$\begin{aligned} \text{PE}_{\text{tot}} &= \int_0^{H+\eta} \rho_0 g z \, dz \, dx = \frac{1}{2} \rho_0 g \int_{-L/2}^{L/2} (H+\eta)^2 \, dx \\ &= \frac{1}{2} \rho_0 g \int_{-L/2}^{L/2} (H^2 + 2H\eta + \eta^2) \, dx = \frac{1}{2} \rho_0 g \left[ Lx H^2 + \int_{-L/2}^{L/2} \eta^2 \, dx \right] \end{aligned}$$

$$\text{PE} = \text{PE}_{\text{tot}} - \frac{1}{2} \rho_0 g H^2 = \frac{1}{2} \rho_0 g \int \eta^2 \, dx$$

its the PE that can be converted  
 to KE  $\rightarrow$  APE (available potential  
 energy)

The total KE is

$$\text{KE} = \frac{1}{2} \rho_0 H \int (u^2 + v^2) \, dx \quad \left[ \text{Assume } H+\eta \approx H \right]$$

The total PE of the system is infinite  
 since the free surface displacement goes off  
 to infinity. But if we look at the  
 change in PE from  $t=0$

$$\Delta \text{PE} = \frac{1}{2} \rho_0 g \int \left( \eta^2 \Big|_{t \rightarrow \infty} - \eta^2 \Big|_{t=0} \right) \, dx$$

$$\begin{aligned}
\Delta PE &= \frac{1}{2} \rho_0 g \left\{ \int_{-\infty}^0 \left[ (1 - e^{x/Lr})^2 - 1 \right] dx + \int_0^{\infty} \left[ (e^{-x/Lr} - 1)^2 - 1 \right] dx \right\} \\
&= \frac{1}{2} \rho_0 g \eta_0^2 \left\{ \int_{-\infty}^0 (e^{2x/Lr} - 2e^{x/Lr}) dx \right. \\
&\quad \left. + \int_0^{\infty} (e^{-2x/Lr} - 2e^{-x/Lr}) dx \right\} \\
&= \frac{1}{2} \rho_0 g \eta_0^2 \left\{ \left[ \frac{Lr}{2} e^{2x/Lr} \right]_{-\infty}^0 - 2Lr e^{x/Lr} \Big|_{-\infty}^0 \right. \\
&\quad \left. + \left[ \frac{Lr}{2} e^{-2x/Lr} \right]_0^{\infty} + 2Lr e^{-x/Lr} \Big|_0^{\infty} \right\} \\
&= \frac{1}{2} \rho_0 g \eta_0^2 \left\{ \frac{Lr}{2} - 2Lr + \frac{Lr}{2} - 2Lr \right\} \\
&= \frac{1}{2} \rho_0 g \eta_0^2 \{ -3 \} = -\frac{3}{2} g \eta_0^2 Lr
\end{aligned}$$

$$\begin{aligned}
KE|_{t \rightarrow \infty} &= \frac{1}{2} \rho_0 \int_{-\infty}^{\infty} v^2 dx \\
&= \frac{1}{2} \rho_0 \frac{\eta_0^2 g^2}{f^2 Lr} \left[ \int_{-\infty}^0 e^{+2x/Lr} dx + \int_0^{\infty} e^{-2x/Lr} dx \right] \\
&= \frac{1}{2} \rho_0 \eta_0^2 g \left[ \frac{Lr}{2} e^{2x/Lr} \Big|_{-\infty}^0 - \frac{Lr}{2} e^{-2x/Lr} \Big|_0^{\infty} \right] \\
&= \frac{1}{2} \rho_0 \eta_0^2 g \left[ \frac{Lr}{2} + \frac{Lr}{2} \right] = \frac{1}{2} \rho_0 \eta_0^2 g Lr
\end{aligned}$$

There are two important things to notice with this solution

① The change in PE is finite. Contrast this to the non-rotating limit where as  $t \rightarrow \infty$  the free surface would flatten completely leading to an infinite change in total PE. In this the non-rotating limit all of this APE goes into generating KE. With rotation not all of the initial PE of the system is converted to KE.

② The change in PE is greater than the KE energy of the balanced geostrophic flow.

Where did the extra PE go?

→ Into transient motions or unbalanced motions.